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Functional relations and analytic Bethe ansatz for twisted quantum affine algebras

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Abstract. Functional relations are proposed for transfer matrices of solvable vertex models associated with the twisted quantum affine algebras $U_q(X_n^{(\kappa)})$, where $X_n^{(\kappa)} = A_n^{(2)}$, $D_n^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$. Their solutions are obtained for $A_n^{(2)}$ and conjectured for $D_4^{(3)}$ in the dressed vacuum form of the analytic Bethe ansatz.

1. Introduction

Solvable lattice models in two dimensions have a commuting family of row-to-row transfer matrices [1]. In [2], a set of functional relations, the T-system, was proposed for the vertex and RSOS models associated with any non-twisted quantum affine algebra $U_q(X_r^{(1)})$. In the QISM terminology [3], the T-system connects the transfer matrices with various fusion types in the auxiliary space but acts on a common quantum space. It generalizes earlier functional relations [4–8] and enables the calculation of various physical quantities [9]. In particular, one can derive eigenvalue formulae for the transfer matrices of fusion vertex models in the dressed vacuum form (DVF) by combining the T-system with the analytic Bethe ansatz. This programme was executed extensively in [10–12], where curious Yangian analogues of the Young tableaux have emerged as natural objects describing the DVFs.

The aim of this paper is to extend these recent works to the case of the twisted quantum affine algebras $U_q(X_n^{(\kappa)})$, where $X_n^{(\kappa)} = A_n^{(2)}$, $D_n^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$. After some preliminaries in the next section, we propose the T-system for solvable vertex models associated with quantum R-matrices of $U_q(X_n^{(\kappa)})$ in section 3. The structure of the T-system is closely analogous to the realization (X_n, σ) of $X_n^{(\kappa)}$ via the classical Lie algebra X_r and the Dynkin diagram automorphism σ of order κ. Except for the 'modulo σ relations', the T-system has the same form as the non-twisted case $X_n^{(1)}$ (see (3.4)). Owing to the commutatibity (see (3.2)), one may regard the T-system as functional relations of the eigenvalues and seek solutions in the DVFs from the analytic Bethe ansatz viewpoint. This is the subject of the subsequent sections. In section 4, we indicate how the solutions can be constructed from those of the non-twisted case $X_n^{(1)}$, in general. As an example, we present the full answer for the $A_n^{(2)}$ T-system in section 5. In section 6, we give a conjectural solution to the $D_4^{(3)}$ T-system. With a slight modification, this also yields a full conjecture for the $D_4^{(1)}$ case. These DVFs are neatly described by Yangian analogues of the Young tableaux [11, 12]. Section 7 is devoted to a summary and discussion.

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2. Twisted quantum affine algebras

Let $X_n^{(\kappa)}$ be one of the twisted affine Lie algebras $A_n^{(2)}$ ($n \geqslant 2$), $D_n^{(2)}$ ($n \geqslant 4$), $E_6^{(2)}$ and $D_4^{(3)}$, which are realized with the pair (X_n, σ) of the classical simple Lie algebra X_n and the Dynkin diagram automorphism σ of order $\kappa = 1, 2$ or 3 [13]. We write the set of the nodes on the diagram as $\mathcal{S} = \{1, 2, \ldots, n\}$ (numeration as in [2]). Then $\sigma : \mathcal{S} \to \mathcal{S}$ is the following map:

$$
\begin{aligned}
&A_n^{(2)} : \sigma(a) = n + 1 - a \qquad a \in \{1, 2, \ldots, n\} \\
&D_n^{(2)} : \sigma(a) = \begin{cases} a & a \in \{1, 2, \ldots, n-2\} \\ n-1 & a = n \\ n & a = n-1 \end{cases} \\
&E_6^{(2)} : \sigma(a) = \begin{cases} 6 - a & a \in \{1, 2, 3, 4, 5\} \\ 6 & a = 6 \end{cases} \\
&D_4^{(3)} : \sigma(1) = 3 \qquad \sigma(2) = 2 \qquad \sigma(3) = 4 \qquad \sigma(4) = 1.
\end{aligned}
\tag{2.1}
$$

Let $\hat{\mathcal{S}} = \{1, 2, \ldots, r\}$ denote the set $\mathcal{S}$ divided by the identification via σ. We explicitly specify the identification map $\wedge : \mathcal{S} \to \hat{\mathcal{S}}$ as follows

$$
\begin{aligned}
&X_n^{(2)} = A_{2r}^{(2)}, A_{2r-1}^{(2)} : \hat{a} = \min(a, n + 1 - a) \\
&X_n^{(2)} = D_{r+1}^{(2)} : \hat{a} = \begin{cases} a & a \in \{1, 2 \ldots, r-1\} \\ r & a = r, r+1 \end{cases} \\
&X_n^{(2)} = E_6^{(2)} (r = 4) : \hat{a} = \begin{cases} \min(a, 6 - a) & a \neq 6 \\ 4 & a = 6 \end{cases} \\
&X_n^{(2)} = D_4^{(3)} (r = 2) : \hat{1} = 1 \qquad \hat{2} = 2 \qquad \hat{3} = 1 \qquad \hat{4} = 1.
\end{aligned}
\tag{2.2a}
$$

We shall keep this relation between n and r throughout. We also introduce the natural embedding $\cdot : \hat{\mathcal{S}} \to \mathcal{S}$ by

$$
\dot{a} = \begin{cases} 6 & \text{if } X_n^{(\kappa)} = E_6^{(2)} \text{ and } a = 4 \\ a & \text{otherwise.} \end{cases}
\tag{2.2b}
$$

Obviously, the composition $\hat{\mathcal{S}} \xrightarrow{\cdot} \mathcal{S} \xrightarrow{\sigma} \mathcal{S} \xrightarrow{\wedge} \hat{\mathcal{S}}$ is the identity and so is the restriction of $\mathcal{S} \xrightarrow{\wedge} \hat{\mathcal{S}} \xrightarrow{\cdot} \mathcal{S}$ on the image of $\hat{\mathcal{S}} \xrightarrow{\cdot} \mathcal{S}$. Consider the quantized universal enveloping algebra $U_q(X_n^{(\kappa)})$ of $X_n^{(\kappa)}$ introduced in [14]. In this paper we assume that the deformation parameter q is generic and put

$$
q = \mathrm{e}^{\hbar}
$$

where $\hbar$ is a parameter. Let $R_{W,W'}(u)$ be the quantum R-matrix [14] that intertwines the tensor products $W \otimes W'$ and $W' \otimes W$ of the finite dimensional irreducible $U_q(X_n^{(\kappa)})$-modules W and W'. It obeys the Yang–Baxter equation [1]

$$
R_{W,W'}(u) R_{W,W''}(u + v) R_{W',W''}(v) = R_{W',W''}(v) R_{W,W''}(u + v) R_{W,W'}(u)
\tag{2.3}
$$

where $u, v \in \mathbb{C}$ are the spectral parameters. $R_{W,W'}(u)$ is a rational function of q^u, which can be properly normalized to be pole-free.

In this paper we shall exclusively consider the family of the irreducible finite dimensional $U_q(X_n^{(\kappa)})$-modules $\{\hat{W}_m^{(a)} \mid a \in \hat{S}, m \in \mathbb{Z}_{\geqslant 1}\}$. If $X_n^{(\kappa)} = A_n^{(2)}$ or $D_n^{(2)}$, the module $\hat{W}_1^{(1)}$ is the affinization of the vector representation for which the associated R-matrix $R_{\hat{W}_1^{(1)},\hat{W}_1^{(1)}}(u)$ has been calculated in [15]. See also the appendix in [16]. For these algebras, $R_{W,W'}(u)$ for the whole family $W, W' \in \{\hat{W}_m^{(a)}\}$ will be obtained by the fusion [17] of the above R-matrix, except for those associated with the spinor-like series $\hat{W}_m^{(r)}$ in $D_{r+1}^{(2)}$. In general, $\hat{W}_m^{(a)}$ is an analogue of the m-fold symmetric tensor representation of $\hat{W}_1^{(a)}$. If one denotes by $W_m^{(a)}$ the irreducible finite dimensional $U_q(X_n^{(1)})$-module sketched in [2], one obtains

$$\dim \hat{W}_m^{(a)} = \dim W_m^{(\dot{a})} \qquad a \in \hat{S}. \tag{2.4}$$

The right-hand side here is essentially the quantity $Q_m^{(\dot{a})}$ in [2, 18] and is computable as a certain sum of the dimensions of irreducible X_n-modules [2, 18, 19]. See also (3.10). Although these descriptions of $\hat{W}_m^{(a)}$ are, in general, conjectural and incomplete, they are consistent with the loop algebra realization of $X_n^{(\kappa)}$ [13] as well as the analytic Bethe ansatz studies in [20] and sections 4, 5 and 6.

3. T-system for twisted quantum affine algebras

Now we turn to the transfer matrix

$$T_m^{(a)}(u) = \mathrm{Tr}_{\hat{W}_m^{(a)}}(R_{\hat{W}_m^{(a)},\hat{W}_s^{(p)}}(u - w_1) \cdots R_{\hat{W}_m^{(a)},\hat{W}_s^{(p)}}(u - w_N)). \tag{3.1}$$

Here N denotes the system size, $w_1, \ldots, w_N$ are complex parameters representing the inhomogeneity, $p \in \hat{S}$ and $s \in \mathbb{Z}_{\geqslant 1}$. The trace (3.1) is the row-to-row transfer matrix with the auxiliary space $\hat{W}_m^{(a)}$ acting on the quantum space $(\hat{W}_s^{(p)})^{\otimes N}$ (more precisely, $\hat{W}_m^{(a)}(u)$ and $\otimes_{j=1}^N \hat{W}_s^{(p)}(w_j)$, respectively.) We have suppressed the quantum-space dependence on the left-hand side of (3.1), reserving the letters p and s for this meaning throughout. Thanks to the Yang–Baxter equation (2.3), the transfer matrices (3.1) form a commuting family

$$[T_m^{(a)}(u), T_{m'}^{(a')}(u')] = 0. \tag{3.2}$$

We shall write the eigenvalues of $T_m^{(a)}(u)$ as $\Lambda_m^{(a)}(u)$. Our purpose here is to propose the T-system, a set of functional relations among the transfer matrices $\{T_m^{(a)}(u) | a \in \hat{S}, m \in \mathbb{Z}_{\geqslant 1}\}$, acting on the common quantum space as above. To do so, we first recall the T-system for the corresponding non-twisted cases $X_n^{(1)} = A_n^{(1)}, D_n^{(1)}, E_6^{(1)}$ and $D_4^{(1)}$ [2]. The T-system in these cases has the simple form ($a \in S, m \in \mathbb{Z}_{\geqslant 1}$):

$$T_m^{(a)}(u+1)T_m^{(a)}(u-1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + g_m^{(a)}(u) \prod_{b \in S_a} T_m^{(b)}(u). \tag{3.3}$$

Here S_a stands for the set of adjacent nodes to $a \in S$ on the X_n-Dynkin diagram. $g_m^{(a)}(u)$ is a scalar function analogous to the quantum determinant and depends on the the choice of the quantum space. It satisfies $g_m^{(a)}(u+1)g_m^{(a)}(u-1) = g_{m+1}^{(a)}(u)g_{m-1}^{(a)}(u)$ [2]. The T-system for $X_n^{(\kappa)}$ is formally obtained from (3.3) by further imposing the 'modulo σ relations':

$$T_m^{(a)}(u) = \epsilon_m^a T_m^{(\sigma(a))}\left(u + \frac{\pi\mathrm{i}}{\kappa\hbar}\right) \tag{3.4a}$$

$$(\epsilon_m^a)^\kappa = (-)^{DN} \tag{3.4b}$$

where D will be explained after (4.3). By means of (3.4a), one can confine (3.3) into equations among only $T_m^{(a)}(u)$ with $a \in \hat{S}$. We call the the resulting functional relation the $X_n^{(\kappa)}$ T-system. Apart from the $a = \sigma(a)$ case in (3.4), it reads as follows ($T_0^{(a)}(u) = T_m^{(0)}(u) = 1$).

$A_{2r}^{(2)}$:

$$\begin{aligned} T_m^{(a)}(u+1)T_m^{(a)}(u-1) &\\ = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) &+ g_m^{(a)}(u)T_m^{(a+1)}(u)T_m^{(a-1)}(u) \qquad 1 \leqslant a \leqslant r-1 \\ T_m^{(r)}(u+1)T_m^{(r)}(u-1) = T_{m+1}^{(r)}(u)T_{m-1}^{(r)}(u) &+ g_m^{(r)}(u)T_m^{(r)}\left(u+\frac{\pi\mathrm{i}}{2\hbar}\right)T_m^{(r-1)}(u). \end{aligned} \tag{3.5}$$

$A_{2r-1}^{(2)}$:

$$\begin{aligned} T_m^{(a)}(u+1)T_m^{(a)}(u-1) &\\ = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) &+ g_m^{(a)}(u)T_m^{(a+1)}(u)T_m^{(a-1)}(u) \qquad 1 \leqslant a \leqslant r-1 \\ T_m^{(r)}(u+1)T_m^{(r)}(u-1) = T_{m+1}^{(r)}(u)T_{m-1}^{(r)}(u) &+ g_m^{(r)}(u)T_m^{(r-1)}\left(u+\frac{\pi\mathrm{i}}{2\hbar}\right)T_m^{(r-1)}(u). \end{aligned} \tag{3.6}$$

$D_{r+1}^{(2)}$:

$$\begin{aligned} T_m^{(a)}(u+1)T_m^{(a)}(u-1) &\\ = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) &+ g_m^{(a)}(u)T_m^{(a+1)}(u)T_m^{(a-1)}(u) \qquad 1 \leqslant a \leqslant r-2 \\ T_m^{(r-1)}(u+1)T_m^{(r-1)}(u-1) &\\ = T_{m+1}^{(r-1)}(u)T_{m-1}^{(r-1)}(u) &+ g_m^{(r-1)}(u)T_m^{(r-2)}(u)T_m^{(r)}(u)T_m^{(r)}\left(u+\frac{\pi\mathrm{i}}{2\hbar}\right) \\ T_m^{(r)}(u+1)T_m^{(r)}(u-1) = T_{m+1}^{(r)}(u)T_{m-1}^{(r)}(u) &+ g_m^{(r)}(u)T_m^{(r-2)}(u). \end{aligned} \tag{3.7}$$

$E_6^{(2)}$:

$$\begin{aligned} T_m^{(1)}(u+1)T_m^{(1)}(u-1) &= T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + g_m^{(1)}(u)T_m^{(2)}(u) \\ T_m^{(2)}(u+1)T_m^{(2)}(u-1) &= T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + g_m^{(2)}(u)T_m^{(1)}(u)T_m^{(3)}(u) \\ T_m^{(3)}(u+1)T_m^{(3)}(u-1) &= T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + g_m^{(3)}(u)T_m^{(2)}(u)T_m^{(2)}\left(u+\frac{\pi\mathrm{i}}{2\hbar}\right)T_m^{(4)}(u) \\ T_m^{(4)}(u+1)T_m^{(4)}(u-1) &= T_{m+1}^{(4)}(u)T_{m-1}^{(4)}(u) + g_m^{(4)}(u)T_m^{(3)}(u). \end{aligned} \tag{3.8}$$

$D_4^{(3)}$:

$$\begin{aligned} T_m^{(1)}(u+1)T_m^{(1)}(u-1) &= T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + g_m^{(1)}(u)T_m^{(2)}(u) \\ T_m^{(2)}(u+1)T_m^{(2)}(u-1) & \\ &= T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + g_m^{(2)}(u)T_m^{(1)}(u)T_m^{(1)}\left(u+\frac{\pi \mathrm{i}}{3\hbar}\right)T_m^{(1)}\left(u-\frac{\pi \mathrm{i}}{3\hbar}\right). \end{aligned} \tag{3.9}$$

In equations (3.5)–(3.9), we have absorbed the ϵ_m^a factors into redefinitions of $g_m^{(a)}(u)$. The $A_2^{(2)}$ case of (3.5) agrees with equation (15) of [5] for the Izergin–Korepin model [21].

Our proposal (3.5)–(3.9) has mainly stemmed from the analytic Bethe ansatz study of the transfer-matrix eigenvalues similar to [12]. By using the $U_q(X_n^{(\kappa)})$-Bethe equation with a 'modulo σ structure' [22] (see (4.2)), one can consistently build the DVFs as in [12] to convince oneself that the eigenvalues $\Lambda_m^{(a)}(u)$ and, hence, $T_m^{(a)}(u)$ will obey (3.4). This will be seen in sections 4, 5 and 6.

If one drops the u-dependence totally by setting $T_m^{(a)}(u+\cdots) = \hat{Q}_m^{(a)}$, $g_m^{(a)}(u) = 1$ and $\epsilon_m^a = 1$, the resulting equations on $\hat{Q}_m^{(a)}$ have a solution

$$\hat{Q}_m^{(a)} = Q_m^{(\dot{a})}|_{\text{specialization}} \qquad a \in \hat{\mathcal{S}}. \tag{3.10}$$

Here $Q_m^{(\dot{a})}$ denotes the Yangian analogue of the character that satisfies the Q-system for X_n [18, 19, 2]. The specialization on the right-hand side is performed so that the σ-invariance $Q_m^{(b)} = Q_m^{(\sigma(b))}(\forall b \in \mathcal{S})$ is achieved. Certainly (3.10) becomes (2.4) under such a specialization where the characters are reduced to the dimensions.

As in section 5 of [2], one can express $T_m^{(a)}(u)$ $(a \in \hat{\mathcal{S}})$ in terms of $T_1^{(a)}(u + \text{shifts})$ by applying the T-system (3.5)–(3.9) recursively. The result is simply related to the non-twisted case [2] via (3.4) with $\epsilon_m^a = 1$.

4. Analytic Bethe ansatz

For $a \in \mathcal{S}$, let α_a and ω_a be the simple root and fundamental weight of $X_n = A_n, D_n, E_6$, respectively. We employ the standard normalization $(\alpha_a|\alpha_a) = 2$, $(\alpha_a|\omega_b) = \delta_{ab}$ via the bilinear form $(\ |\ )$. Define the functions

$$[u]_k = q^{ku} - q^{-ku} \tag{4.1a}$$

$$\phi^k(u) = \prod_{j=1}^{N}[u - w_j]_k \tag{4.1b}$$

$$Q_a^k(u) = \prod_{j=1}^{N_a}[u - \mathrm{i}u_j^{(a)}]_k \qquad \text{for } a \in \hat{\mathcal{S}} \tag{4.1c}$$

where N_a is a non-negative integer. When $k = 1$, we simply write (4.1a) and (4.1b) as $\phi(u) = \phi^1(u)$ and $Q_a(u) = Q_a^1(u)$. There should be no confusion between $Q_a^k(u)$ here and

$Q_m^{(\dot a)}$ in (3.10). In this notation, the $U_q(X_n^{(\kappa)})$-Bethe-ansatz equation (BAE) [22] is given as follows.

$$-\prod_{t=0}^{\kappa-1}\frac{\phi(\mathrm{i}u_j^{(a)}+(s\omega_{\dot p}|\alpha_{\sigma^t(\dot a)})+(t\pi\mathrm{i}/\kappa\hbar))}{\phi(iu_j^{(a)}-(s\omega_{\dot p}|\alpha_{\sigma^t(\dot a)})+(t\pi\mathrm{i}/\kappa\hbar))}$$
$$=\prod_{t=0}^{\kappa-1}\prod_{b\in\hat{\mathcal{S}}}\frac{Q_b(iu_j^{(a)}+(\alpha_{\dot a}|\alpha_{\sigma^t(\dot b)})+(t\pi\mathrm{i}/\kappa\hbar))}{Q_b(iu_j^{(a)}-(\alpha_{\dot a}|\alpha_{\sigma^t(\dot b)})+(t\pi\mathrm{i}/\kappa\hbar))}. \tag{4.2}$$

For any quantum-space choice labelled by $p\in\hat{\mathcal{S}}$ and $s\in\mathbb{Z}_{\geqslant 1}$, this is a system of simultaneous equations on the complex numbers $\{u_j^{(a)}|a\in\hat{\mathcal{S}},1\leqslant j\leqslant N_a\}$. The $Q_a^k(u)$ (4.1c) are defined for each solution to the above BAE.

In the rest of this section we briefly observe a few features in the analytic Bethe ansatz [23] based on (4.2). This method is a hypothesis that the transfer-matrix eigenvalue is expressed in the DVF:

$$\Lambda_m^{(a)}(u)=\sum_{h=1}^{\dim\hat W_m^{(a)}}T_h \tag{4.3a}$$

$$T_h=(\mathrm{dr}\,T_h)(\mathrm{vac}\,T_h) \tag{4.3b}$$

$$\mathrm{dr}\,T_h=\frac{Q_{a_1^h}^{k_1^h}\left(u+x_1^h\right)\cdots Q_{a_{j(h)}^h}^{k_{j(h)}^h}\left(u+x_{j(h)}^h\right)}{Q_{a_1^h}^{k_1^h}\left(u+y_1^h\right)\cdots Q_{a_{j(h)}^h}^{k_{j(h)}^h}\left(u+y_{j(h)}^h\right)} \tag{4.3c}$$

$$\mathrm{vac}\,T_h=\phi^{l_1^h}\left(u+z_1^h\right)\cdots\phi^{l_{d(h)}^h}\left(u+z_{d(h)}^h\right). \tag{4.3d}$$

Here, (4.3c) and (4.3d) are called the dress part and the vacuum part, respectively. $d(h), j(h), k_i^h, l_i^h, a_i^h, x_i^h, y_i^h$ and z_i^h are to be determined so that (4.3) fulfils a couple of conditions inherited from the properties of the relevant R-matrix $R_{\hat W_m^{(a)},\hat W_s^{(p)}}(u)$ [23, 12]. In particular, they are to be chosen so that $\Lambda_m^{(a)}(u)$ becomes pole-free on condition that the BAE (4.2) is valid. The sum $D=l_1^h+\cdots+l_{d(h)}^h$ is independent of h. Let

$$\sum_{h=1}^{\dim W_m^{(\dot a)}}\dot T_h \tag{4.4a}$$

$$\dot T_h=(\mathrm{dr}\,\dot T_h)(\mathrm{vac}\,\dot T_h) \tag{4.4b}$$

be the DVF for the eigenvalue of the transfer matrix $T_m^{(\dot a)}(u)$ for the $X_n^{(1)}$ case. This corresponds to the non-twisted counterpart of (4.3) in the light of (2.4). In (4.4a), the dress part $\mathrm{dr}\,\dot T_h$ is a ratio of the $X_n^{(1)}$ analogue of $Q_a^1(u)$ (4.1c), which are determined from the solutions to the BAE (4.2) for the non-twisted case $\kappa=1, \mathcal{S}=\hat{\mathcal{S}}$. Let us write this as $\dot Q_a(u)$ ($a\in\mathcal{S}$). Because of (2.4), the sums (4.3a) and (4.4a) consist of the same number of terms. Moreover, one can relate the dress parts as

$$\mathrm{dr}\,T_h=\mathrm{dr}\,\dot T_h|_{\sigma\text{-reduction}}$$
$$\sigma\text{-reduction}:\ \dot Q_a(u)\to\prod_{t=0}^{\kappa-1}\left(Q_{\hat a}\left(u-\frac{t\pi\mathrm{i}}{\kappa\hbar}\right)\right)^{\nu(a,t)}\qquad\text{for all } a\in\mathcal{S} \tag{4.5}$$
$$\nu(a,t)=\begin{cases}1 & \text{if } \sigma^t(a)=\dot{\hat a}\\ 0 & \text{otherwise.}\end{cases}$$

This follows simply by comparing (4.2) with the non-twisted case $\kappa = 1$. It has already been used implicitly in [20] and will actually be seen in sections 5 and 6.

Let us include a remark before closing this section. Suppose one has found the DVF when the quantum space is $\otimes_{j=1}^{N} W_1^{(p)}(w_j)$. Then, the DVF for $\otimes_{j=1}^{N} W_s^{(p)}(w_j)$ can be deduced from it by the replacement

$$\phi^k(u) \to \prod_{j=1}^{s} \phi^k(u+s+1-2j)$$

(see (4.1) and (4.2)). We shall henceforth consider the $s = 1$ case only, with no loss of generality.

5. Solutions for $A_n^{(2)}$

Here we give the solution of the T-system (3.5), (3.6) for $A_n^{(2)} = A_{2r}^{(2)}$ and $A_{2r-1}^{(2)}$. Define a set J by

$$J = \begin{cases} \{1, 2, \ldots, r, \bar{r}, \ldots, \bar{2}, \bar{1}\} & \text{for } A_{2r-1}^{(2)} \\ \{1, 2, \ldots r, 0, \bar{r}, \ldots, \bar{2}, \bar{1}\} & \text{for } A_{2r}^{(2)} \end{cases} \tag{5.1}$$

and specify the ordering among these letters as

$$\begin{aligned} &1 \prec 2 \prec \cdots \prec r \prec \bar{r} \prec \cdots \prec \bar{2} \prec \bar{1} \qquad \text{for } A_{2r-1}^{(2)} \\ &1 \prec 2 \prec \cdots \prec r \prec 0 \prec \bar{r} \prec \cdots \prec \bar{2} \prec \bar{1} \qquad \text{for } A_{2r}^{(2)}. \end{aligned} \tag{5.2}$$

We introduce the boxes that contain an element of J and represent a dressed vacuum as follows

$$\begin{aligned}
&\boxed{a} = \psi_a(u)\frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)} \qquad 1 \leqslant a \leqslant \left[\frac{n}{2}\right] \\
&\boxed{\bar{a}} = \psi_{\bar{a}}(u)\frac{Q_{a-1}(u+n-a+\frac{\pi i}{2\hbar})Q_a(u+n-a+3+\frac{\pi i}{2\hbar})}{Q_{a-1}(u+n-a+2+\frac{\pi i}{2\hbar})Q_a(u+n-a+1+\frac{\pi i}{2\hbar})} \qquad 1 \leqslant a \leqslant \left[\frac{n}{2}\right] \\
&\boxed{r} = \psi_r(u)\frac{Q_{r-1}(u+r+1)Q_r^2(u+r-2)}{Q_{r-1}(u+r-1)Q_r^2(u+r)} \qquad \text{for } A_{2r-1}^{(2)} \\
&\boxed{\bar{r}} = \psi_{\bar{r}}(u)\frac{Q_{r-1}(u+r-1+\frac{\pi i}{2\hbar})Q_r^2(u+r+2)}{Q_{r-1}(u+r+1+\frac{\pi i}{2\hbar})Q_r^2(u+r)} \qquad \text{for } A_{2r-1}^{(2)} \\
&\boxed{0} = \psi_0(u)\frac{Q_r(u+r+2)Q_r(u+r-1+\frac{\pi i}{2\hbar})}{Q_r(u+r)Q_r(u+r+1+\frac{\pi i}{2\hbar})} \qquad \text{for } A_{2r}^{(2)}
\end{aligned} \tag{5.3a}$$

where we have put $Q_0(u) = 1$. The symbol $[x]$ stands for the largest integer not exceeding x. In the above, the spectral parameter u is attached to each box. We shall exhibit the u-dependence as $\boxed{a}_u$ when necessary. The functions $\psi_a(u)$ are the vacuum parts and depend on the choice of the quantum space. For $s=1$, $1 \leqslant p \leqslant r$, they are given by

$$
\begin{aligned}
\psi_1(u) = \cdots = \psi_p(u) &= \phi(u+p+1)\phi\left(u+n+2-p+\frac{\pi\mathrm{i}}{2\hbar}\right) \\
\psi_{p+1}(u) = \cdots = \psi_{\overline{p+1}}(u) &= \phi(u+p-1)\phi\left(u+n+2-p+\frac{\pi\mathrm{i}}{2\hbar}\right) \qquad (5.3b)\\
\psi_{\overline{p}}(u) = \cdots = \psi_{\overline{1}}(u) &= \phi(u+p-1)\phi\left(u+n-p+\frac{\pi\mathrm{i}}{2\hbar}\right).
\end{aligned}
$$

Note that the second possibility is void for $A^{(2)}_{2r-1}$ and $p=r$.

Now we introduce the set $\mathcal{T}^{(a)}_m$ of the semi-standard tableaux with $a \times m$ rectangular shape:

$$
\mathcal{T}^{(a)}_m = \left\{ \left. \begin{array}{|c|c|c|} \hline i_{11} & \cdots & i_{1m} \\ \hline \vdots & \ddots & \vdots \\ \hline i_{a1} & \cdots & i_{am} \\ \hline \end{array} \right| \; i_{jk} \in J,\ i_{1k} \prec \cdots \prec i_{ak},\ i_{j1} \preceq \cdots \preceq i_{jm} \right\}. \qquad (5.4)
$$

We identify each element of $\mathcal{T}^{(a)}_m$ as above with the product

$$
\prod_{j=1}^{a}\prod_{k=1}^{m} \boxed{i_{jk}}_{\,u+a-m-2j+2k}. \qquad (5.5)
$$

Put

$$
\Lambda^{(a)}_m(u) = \frac{1}{f^{(a)}_m(u)} \sum_{T \in \mathcal{T}^{(a)}_m} T \qquad (5.6a)
$$

$$
f^{(a)}_m(u) = \prod_{j=1}^{m} f^{(a)}_1(u+m+1-2j) \qquad (5.6b)
$$

$$
f^{(a)}_1(u) = \left(\prod_{j=1}^{|a-p|} \phi(u+|a-p|-2j)\phi\left(u-|a-p|+n+1+2j+\frac{\pi\mathrm{i}}{2\hbar}\right)\right)^{\pm 1} \qquad (5.6c)
$$

where the power in (5.6c) should be chosen as $+1$ or -1 according to $a \geqslant p$ or $a < p$, respectively. One can check that for $a > p$, the denominator $f^{(a)}_m(u)$ in (5.6a) can be completely cancelled out for any $T \in \mathcal{T}^{(a)}_m$. Consequently, the vacuum parts in $\Lambda^{(a)}_m(u)$ are homogeneous of order mp with respect to both $\phi(u+\xi)$ and $\phi(u+\eta+\frac{\pi\mathrm{i}}{2\hbar})$, where $\xi, \eta \in \mathbb{Z}$. Now the main claim in this section is the following theorem.

Theorem. For $1 \leqslant p \leqslant r$, the above $\Lambda^{(a)}_m(u)$ is pole-free provided that BAE (4.2) is valid. It satisfies the $A^{(2)}_n$ T-system (3.5), (3.6) with

$$
\begin{aligned}
g^{(a)}_m(u) &= 1 \qquad \text{for } 1 \leqslant a \leqslant r-1 \\
g^{(r)}_m(u) &= (-)^{Nmp}.
\end{aligned} \qquad (5.7)
$$

Owing to (4.5), the proof essentially reduces to that for the $A_n^{(1)}$ case, which can be obtained by combining the results in [7] and [2]. Based on the analytic Bethe ansatz, we thus suppose that (5.6) gives the eigenvalues of the fusion $A_n^{(2)}$ vertex models. For $p = 1$, $\Lambda_1^{(1)}(u)$ actually coincides with the DVF obtained earlier in [20]. For $A_{2r-1}^{(2)}$, the property

$$\Lambda_m^{(r)}(u) = (-)^{Nmp}\Lambda_m^{(r)}\left(u + \frac{\pi \mathrm{i}}{2\hbar}\right)$$

holds, which corresponds to (3.4) for the fixed node $r = \sigma(r)$.

6. Conjectures for $D_4^{(3)}$

A peculiarity of $D_4^{(3)}$ is that this is the only twisted affine Lie algebra with $\kappa = 3$. In this section, we will give conjectures on the solutions to T-system (3.9) for this algebra. First, we define a set J

$$J = \{1, 2, 3, 4, \overline{4}, \overline{3}, \overline{2}, \overline{1}\} \tag{6.1}$$

and specify an order in J as

$$1 \prec 2 \prec 3 \prec \begin{matrix} 4 \\ \overline{4} \end{matrix} \prec \overline{3} \prec \overline{2} \prec \overline{1}. \tag{6.2}$$

We assume no order between 4 and $\overline{4}$. These letters have an origin in the labels for D_4 [12]. As in the $A_n^{(2)}$ case, we introduce elementary boxes carrying $a \in J$ and the spectral parameter u by

$$\begin{aligned}
\boxed{1} &= \psi_1(u)\frac{Q_1(u-1)}{Q_1(u+1)} \\
\boxed{2} &= \psi_2(u)\frac{Q_1(u+3)Q_2^3(u)}{Q_1(u+1)Q_2^3(u+2)} \\
\boxed{3} &= \psi_3(u)\frac{Q_1(u+1+\frac{\pi \mathrm{i}}{3\hbar})Q_1(u+1-\frac{\pi \mathrm{i}}{3\hbar})Q_2^3(u+4)}{Q_1(u+3+\frac{\pi \mathrm{i}}{3\hbar})Q_1(u+3-\frac{\pi \mathrm{i}}{3\hbar})Q_2^3(u+2)} \\
\boxed{4} &= \psi_4(u)\frac{Q_1(u+1-\frac{\pi \mathrm{i}}{3\hbar})Q_1(u+5+\frac{\pi \mathrm{i}}{3\hbar})}{Q_1(u+3-\frac{\pi \mathrm{i}}{3\hbar})Q_1(u+3+\frac{\pi \mathrm{i}}{3\hbar})} \\
\boxed{\overline{4}} &= \psi_{\overline{4}}(u)\frac{Q_1(u+1+\frac{\pi \mathrm{i}}{3\hbar})Q_1(u+5-\frac{\pi \mathrm{i}}{3\hbar})}{Q_1(u+3+\frac{\pi \mathrm{i}}{3\hbar})Q_1(u+3-\frac{\pi \mathrm{i}}{3\hbar})} \\
\boxed{\overline{3}} &= \psi_{\overline{3}}(u)\frac{Q_1(u+5+\frac{\pi \mathrm{i}}{3\hbar})Q_1(u+5-\frac{\pi \mathrm{i}}{3\hbar})Q_2^3(u+2)}{Q_1(u+3+\frac{\pi \mathrm{i}}{3\hbar})Q_1(u+3-\frac{\pi \mathrm{i}}{3\hbar})Q_2^3(u+4)} \\
\boxed{\overline{2}} &= \psi_{\overline{2}}(u)\frac{Q_1(u+3)Q_2^3(u+6)}{Q_1(u+5)Q_2^3(u+4)} \\
\boxed{\overline{1}} &= \psi_{\overline{1}}(u)\frac{Q_1(u+7)}{Q_1(u+5)}.
\end{aligned} \tag{6.3a}$$

In the above, $\psi_a(u)$ denotes the vacuum part and depends on the choice of quantum space. For $s = 1$ and $p = 1, 2$ they read:

$p = 1$:

$$
\begin{aligned}
\psi_1(u) &= \phi(u+2)\phi(u+6)\phi\left(u+4+\frac{\pi i}{3\hbar}\right)\phi\left(u+4-\frac{\pi i}{3\hbar}\right) \\
\psi_2(u) &= \psi_3(u) = \phi(u)\phi(u+6)\phi\left(u+4+\frac{\pi i}{3\hbar}\right)\phi\left(u+4-\frac{\pi i}{3\hbar}\right) \\
\psi_4(u) &= \phi(u)\phi(u+6)\phi\left(u+2+\frac{\pi i}{3\hbar}\right)\phi\left(u+4-\frac{\pi i}{3\hbar}\right) \\
\psi_{\overline{4}}(u) &= \phi(u)\phi(u+6)\phi\left(u+4+\frac{\pi i}{3\hbar}\right)\phi\left(u+2-\frac{\pi i}{3\hbar}\right) \\
\psi_{\overline{3}}(u) &= \psi_{\overline{2}}(u) = \phi(u)\phi(u+6)\phi\left(u+2+\frac{\pi i}{3\hbar}\right)\phi\left(u+2-\frac{\pi i}{3\hbar}\right) \\
\psi_{\overline{1}}(u) &= \phi(u)\phi(u+4)\phi\left(u+2+\frac{\pi i}{3\hbar}\right)\phi\left(u+2-\frac{\pi i}{3\hbar}\right)
\end{aligned}
\tag{6.3b}
$$

$p = 2$:

$$
\begin{aligned}
\psi_1(u) &= \psi_2(u) = \phi^3(u+3)\phi^3(u+5) \\
\psi_3(u) &= \psi_{\overline{3}}(u) = \psi_4(u) = \psi_{\overline{4}}(u) = \phi^3(u+1)\phi^3(u+5) \\
\psi_{\overline{2}}(u) &= \psi_{\overline{1}}(u) = \phi^3(u+1)\phi^3(u+3).
\end{aligned}
$$

Next we consider the following sets $\mathcal{T}_m^{(1)}$ and $\mathcal{T}_m^{(2)}$ of the tableaux containing the elements of J:

$$\mathcal{T}_m^{(1)} = \left\{ \boxed{i_1} \boxed{\cdots} \boxed{i_m} \;\middle|\; i_k \preceq i_{k+1} \text{ and } (i_k, i_{k+1}) \neq (4, \overline{4}), (\overline{4}, 4) \text{ for } 1 \leqslant k < m \right\}$$

$$\mathcal{T}_m^{(2)} = \left\{ \begin{array}{|c|c|c|} \hline i_{11} & \cdots & i_{1m} \\ \hline i_{21} & \cdots & i_{2m} \\ \hline \end{array} \;\middle|\; \text{conditions (i)–(iv) below} \right\}$$

(i) both rows belong to $\mathcal{T}_m^{(1)}$

(ii) $i_{1k} \prec i_{2k}$ or $(i_{1k}, i_{2k}) = (4, \overline{4}), (\overline{4}, 4)$ for $1 \leqslant k \leqslant m$

(iii) the columns $\begin{array}{|c|} \hline 3 \\ \hline 4 \\ \hline \end{array}$ and $\begin{array}{|c|} \hline 4 \\ \hline \overline{3} \\ \hline \end{array}$ do not appear simultaneously (6.4)

(iv) the columns $\begin{array}{|c|} \hline 3 \\ \hline \overline{4} \\ \hline \end{array}$ and $\begin{array}{|c|} \hline \overline{4} \\ \hline \overline{3} \\ \hline \end{array}$ do not appear simultaneously.

We identify elements in $\mathcal{T}_m^{(1)}$ and $\mathcal{T}_m^{(2)}$ with the dressed vacuums by the same rule as (5.5) but with the elementary boxes defined by (6.3).

For $a = 1, 2$, we now put

$$\Lambda_m^{(a)}(u) = \left(\frac{1}{g_m^{(1)}(u)}\right)^{\delta_{a2}} \sum_{T \in \mathcal{T}_m^{(a)}} T \tag{6.5a}$$

$$g_m^{(a)}(u) = \prod_{j=1}^{m} g_1^{(a)}(u+m+1-2j) \tag{6.5b}$$

$$g_1^{(a)}(u) = \begin{cases} \phi^{2a-1}(u-1)\phi^{2a-1}(u+7) & \text{if } a=p \\ 1 & \text{otherwise.} \end{cases} \tag{6.5c}$$

One can show that the denominator $g_m^{(1)}(u)$ in (6.5*a*) can be completely cancelled for any $T \in \mathcal{T}_m^{(2)}$.

Based on an extensive computer check, we have the following conjecture.

Conjecture. The $\Lambda_m^{(a)}(u)$ (6.5) is pole-free provided that BAE (4.2) is valid. It satisfies the $D_4^{(3)}$ T-system (3.9).

A few remarks are in order. First, the DVF for $\Lambda_1^{(1)}(u)$ with $p=1$ agrees with the DVF in [20], which is actually pole-free. Second, the pole freeness can be checked directly for $\Lambda_1^{(a)}(u), a, p \in \{1,2\}$ from the explicit form (6.5). Third, the property

$$\Lambda_1^{(2)}\left(u + \frac{\pi \mathrm{i}}{3\hbar}\right) = \Lambda_1^{(2)}(u)$$

holds, which is consistent with (3.4*a*) for the fixed node $2 = \sigma(2)$. Finally, one can set up a similar conjecture to the one above for the $D_4^{(1)}$ T-system [2], by using (4.5) and the same sets $\mathcal{T}_m^{(a)}$ as in (6.4).

7. Summary and discussion

In this paper we have proposed the T-system, the transfer-matrix functional relations, for the fusion vertex models associated with the twisted quantum affine algebras $U_q(X_n^{(\kappa)})$ for $X_n^{(\kappa)} = A_n^{(2)}, D_n^{(2)}, E_6^{(2)}$ and $D_4^{(3)}$. It extends the non-twisted case in [2, 9]. We have also constructed the DVFs that satisfy or conjecturally satisfy the $A_n^{(2)}$ or $D_4^{(3)}$ T-system, respectively. This is yet a further result following the scheme of [12]. These DVFs have been derived essentially from the non-twisted cases by noting the σ-reduction relation (4.5). Thus, a similar DVF construction will also be possible for the $D_n^{(2)}$ case based on the $D_n^{(1)}$ result in [12]. In view of the analytic Bethe ansatz, these DVFs are the candidates of the transfer-matrix eigenvalues for the fusion vertex models associated with $U_q(X_n^{(\kappa)})$. The T-system here will work efficiently for computing the physical quantities as in [9]. We leave this for a future study.

There are some issues that deserve further investigation. The T-system proposed in this paper applies to the vertex models, which is the *unrestricted* version in the sense of [2]. It will be interesting to introduce the level parameter ℓ and seek the *restricted* T-system that applies to the level ℓ RSOS models, as performed in [2]. In this case, the level–rank duality between the level ℓ $A_{2r-1}^{(2)}$ RSOS model and level r $C_\ell^{(1)}$ RSOS model [16] will be a good guide.

One may set, for example, in the $D_4^{(3)}$ case

$$\begin{aligned} y_m^{(1)}(u) &= \frac{g_m^{(1)}(u)\Lambda_m^{(2)}(u)}{\Lambda_{m+1}^{(1)}(u)\Lambda_{m-1}^{(1)}(u)} \\ y_m^{(2)}(u) &= \frac{g_m^{(2)}(u)\Lambda_m^{(1)}(u)\Lambda_m^{(1)}(u+\frac{\pi \mathrm{i}}{3\hbar})\Lambda_m^{(1)}(u-\frac{\pi \mathrm{i}}{3\hbar})}{\Lambda_{m+1}^{(2)}(u)\Lambda_{m-1}^{(2)}(u)}. \end{aligned} \tag{7.1}$$

Then the T-system (3.9) for the eigenvalues is transformed into

$$
\begin{aligned}
y_m^{(1)}(u+1)y_m^{(1)}(u-1) &= \frac{1+y_m^{(2)}(u)}{(1+y_{m+1}^{(1)}(u)^{-1})(1+y_{m-1}^{(1)}(u)^{-1})} \\
y_m^{(2)}(u+1)y_m^{(2)}(u-1) &= \frac{(1+y_m^{(1)}(u))(1+y_m^{(1)}(u+\frac{\pi i}{3\hbar}))(1+y_m^{(1)}(u-\frac{\pi i}{3\hbar}))}{(1+y_{m+1}^{(2)}(u)^{-1})(1+y_{m-1}^{(2)}(u)^{-1})}
\end{aligned}
\tag{7.2}
$$

which is independent of $g_m^{(a)}(u)$. Equations like (7.2) are called the Y-system in [2,9]. It can be derived similarly for the other cases $A_n^{(2)}$, $D_n^{(2)}$ and $E_6^{(2)}$. For the non-twisted case, the Y-system for general $X_r^{(1)}$ was extracted in [24] from the thermodynamic Bethe ansatz (TBA) equation in [19]. Its structure reflects the string hypothesis employed [19]. It will be interesting to examine whether the Y-system obtained here indicates an appropriate way of setting up the string hypothesis and performing TBA for $U_q(X_n^{(\kappa)})$ symmetry models.

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